

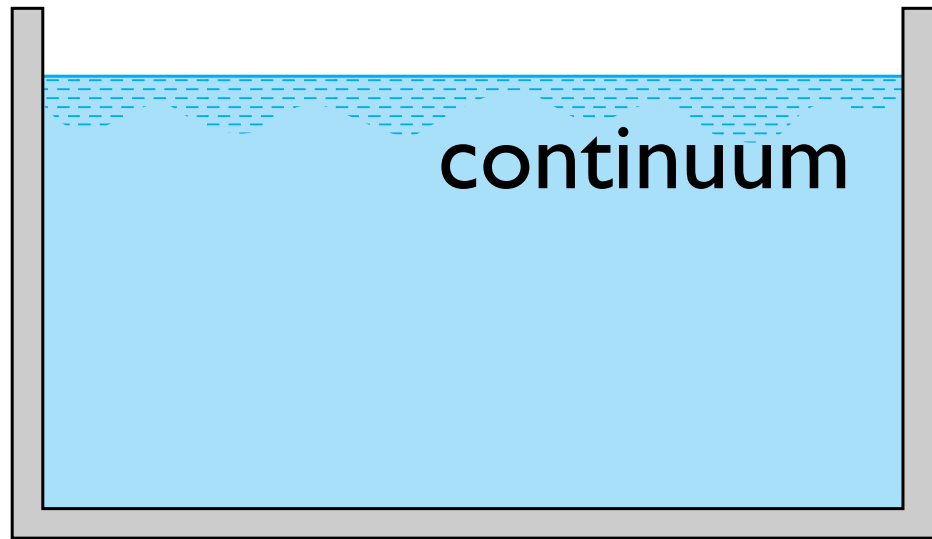
Pressure Integration of the Navier-Stokes Equations

Tony Saad

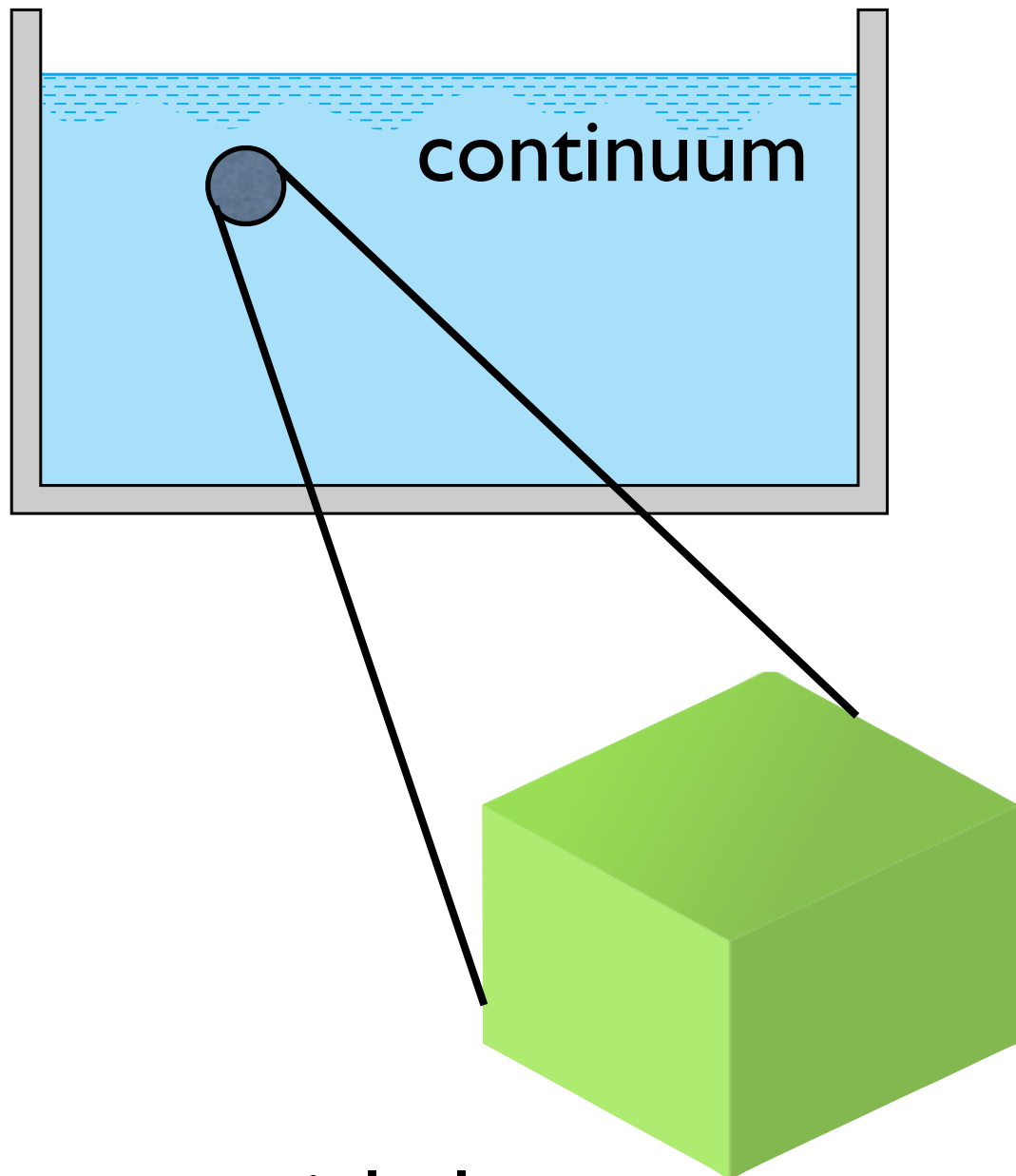
<http://www.tsaad.net>

(Sept 28, 2010)

Scales



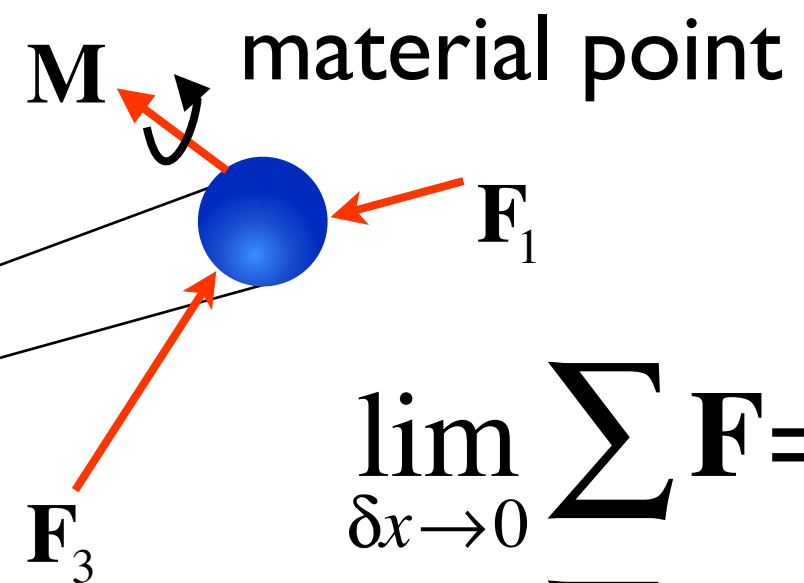
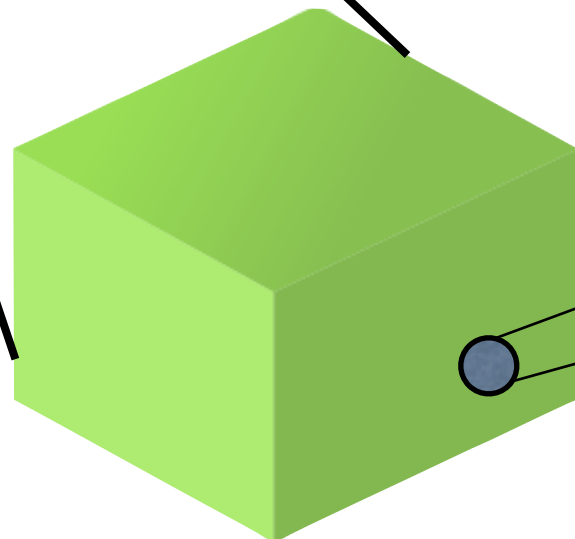
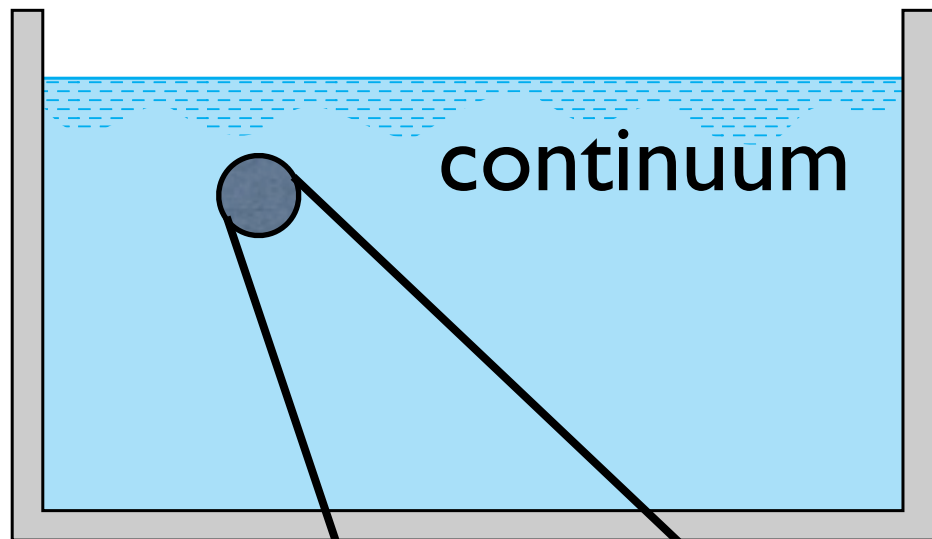
Scales



material element

$$\sum \mathbf{F} = m\mathbf{a}$$

Scales

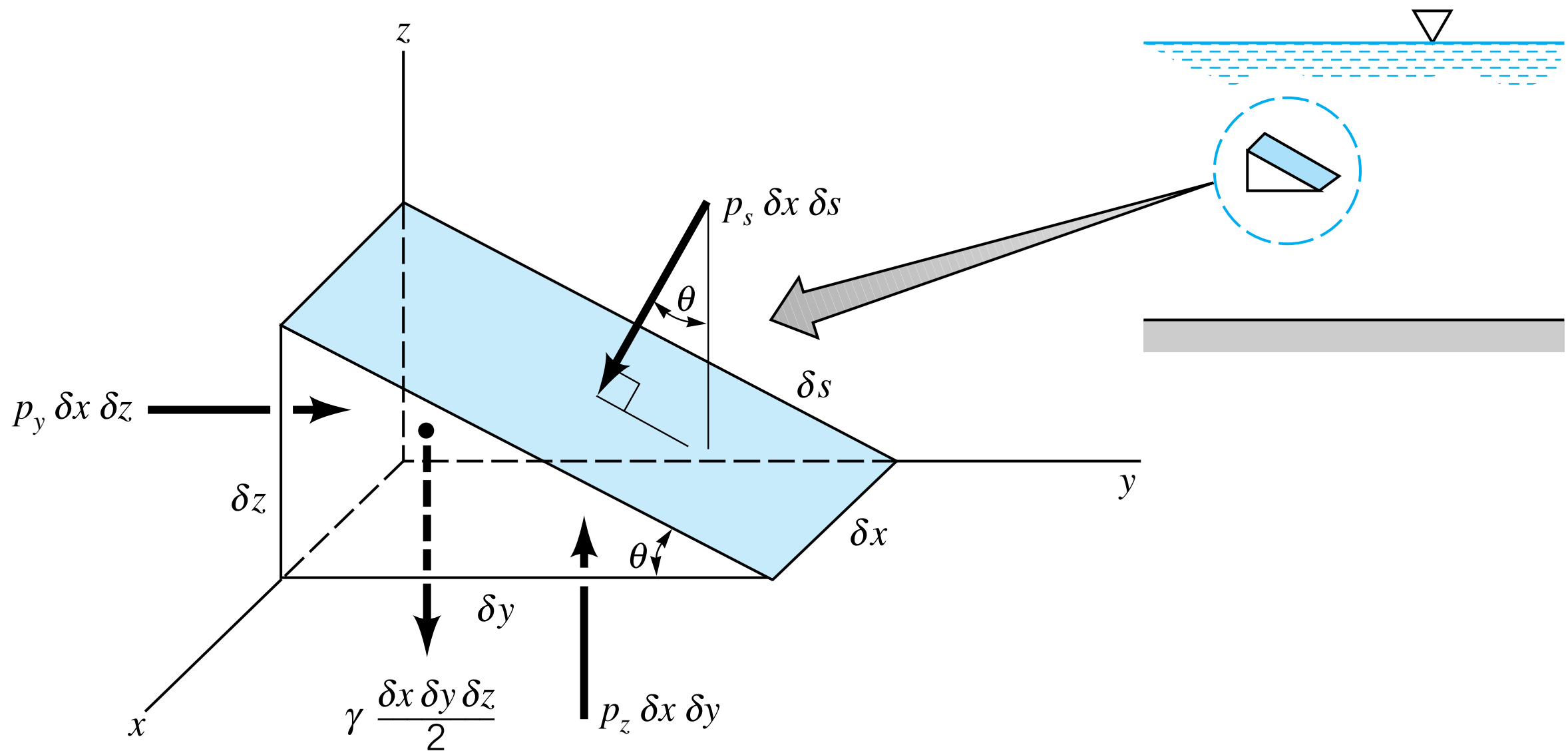


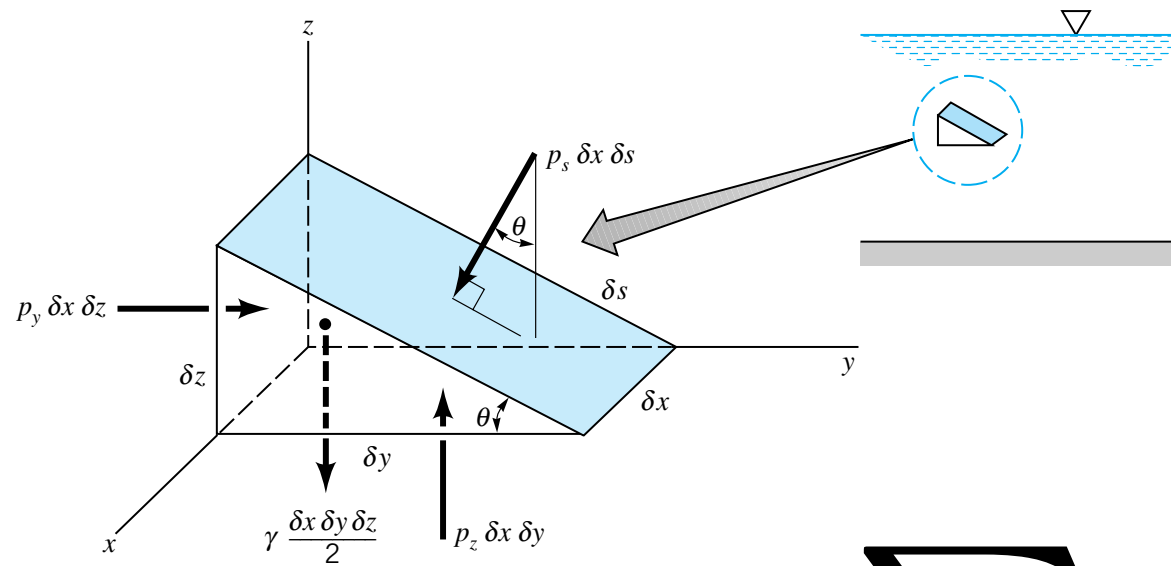
material element

$$\sum \mathbf{F} = m\mathbf{a}$$

$$\lim_{\delta x \rightarrow 0} \sum \mathbf{F} = \mathbf{0}$$
$$\lim_{\delta x \rightarrow 0} \sum \mathbf{M} = \mathbf{0}$$

Pressure at a Point





$$\sum \mathbf{F} = m \mathbf{a}$$

(assume material point is
NOT in static equilibrium)

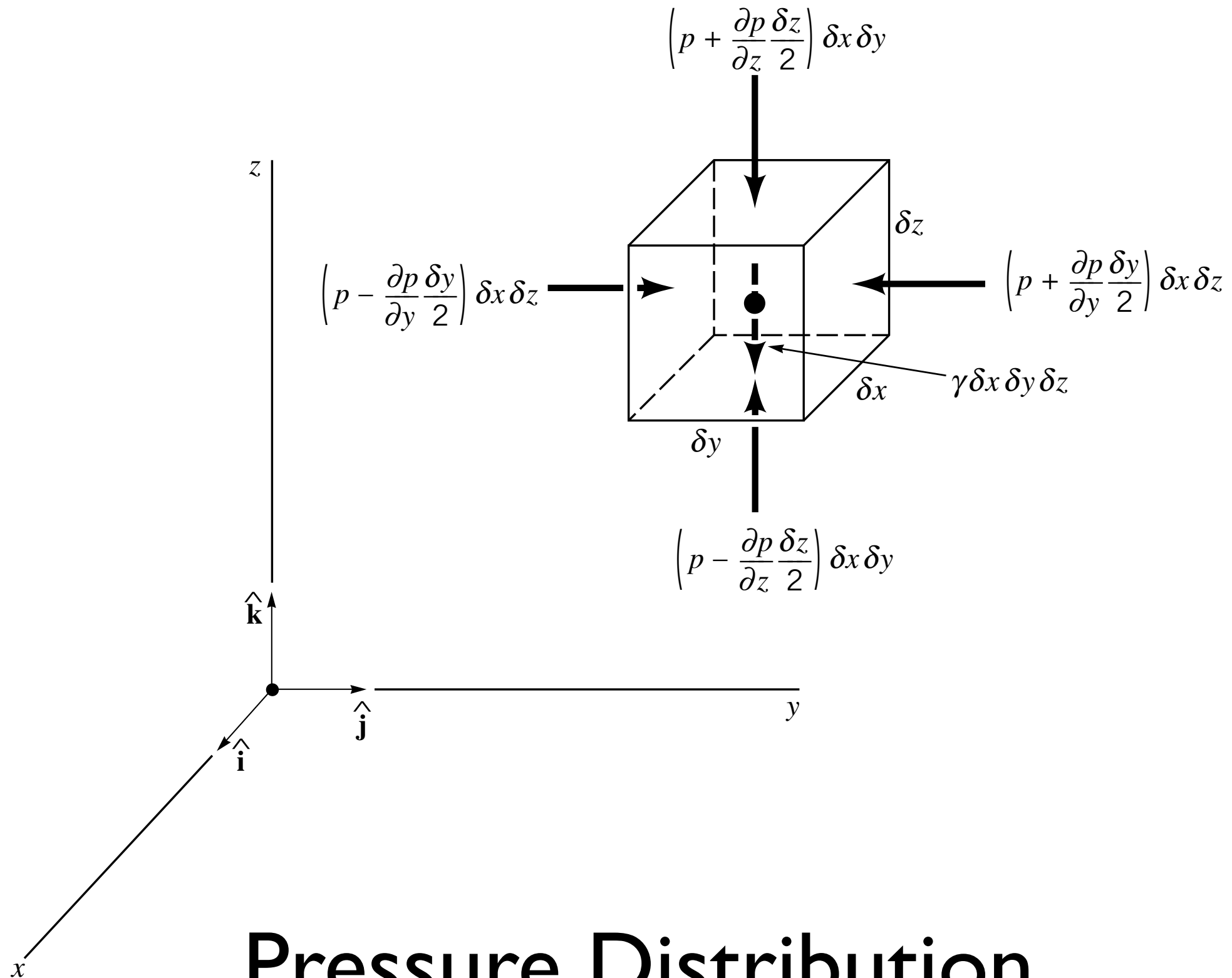
$$p_y \delta x \delta z - p_s \delta x \delta s \sin \theta = \frac{1}{2} \rho \delta x \delta y \delta z a_y$$

$$p_y \delta x \delta z - p_s \delta x \delta z = \frac{1}{2} \rho g \delta x \delta y \delta z a_y$$

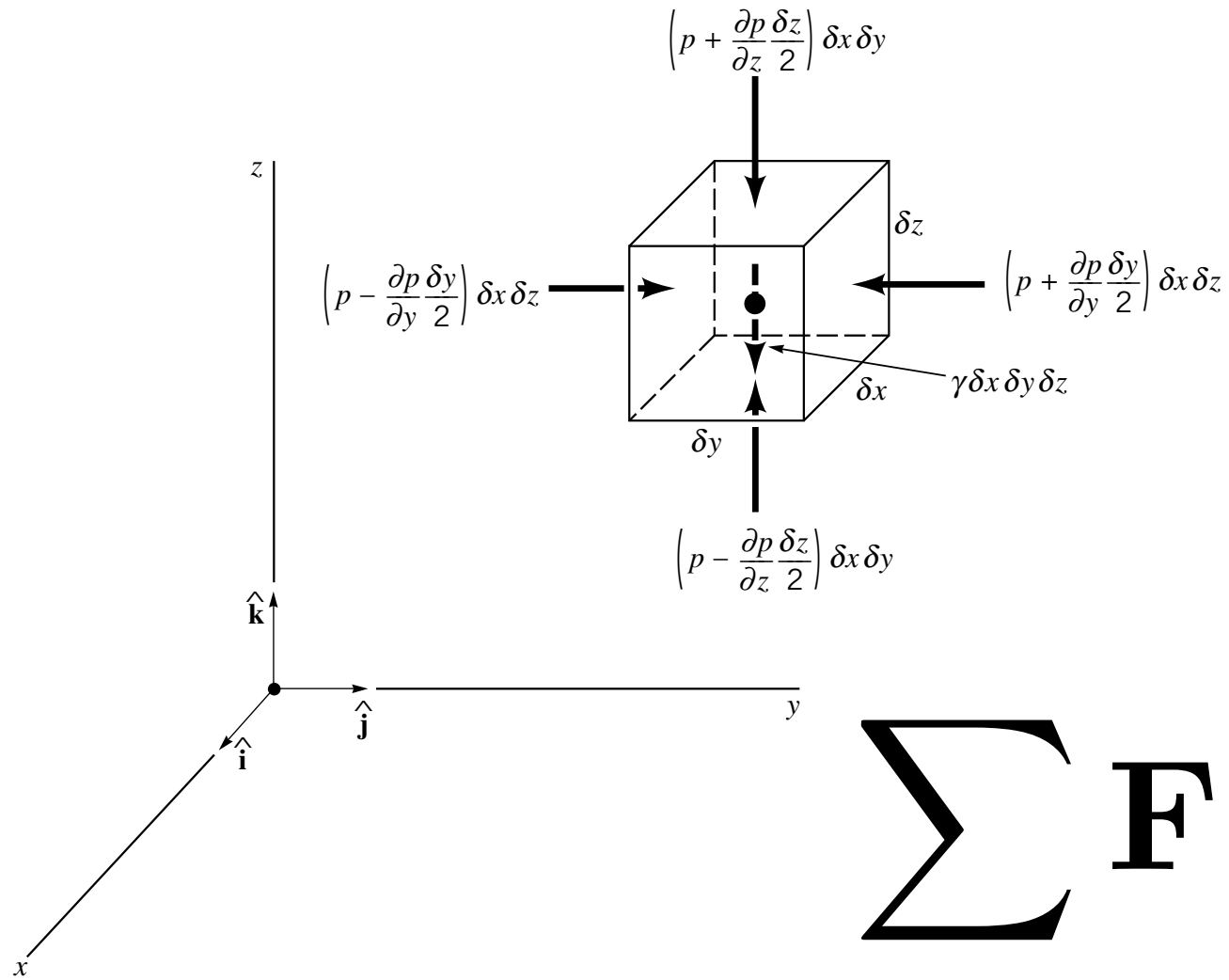
$$p_y - p_s = \frac{1}{2} \rho g \delta y a_y \rightarrow 0$$

$$p_y = p_s$$

**The Pressure at a point
in a static fluid or inviscid
flow (no shear stress) is
equal in all directions**



Pressure Distribution (on a material element)



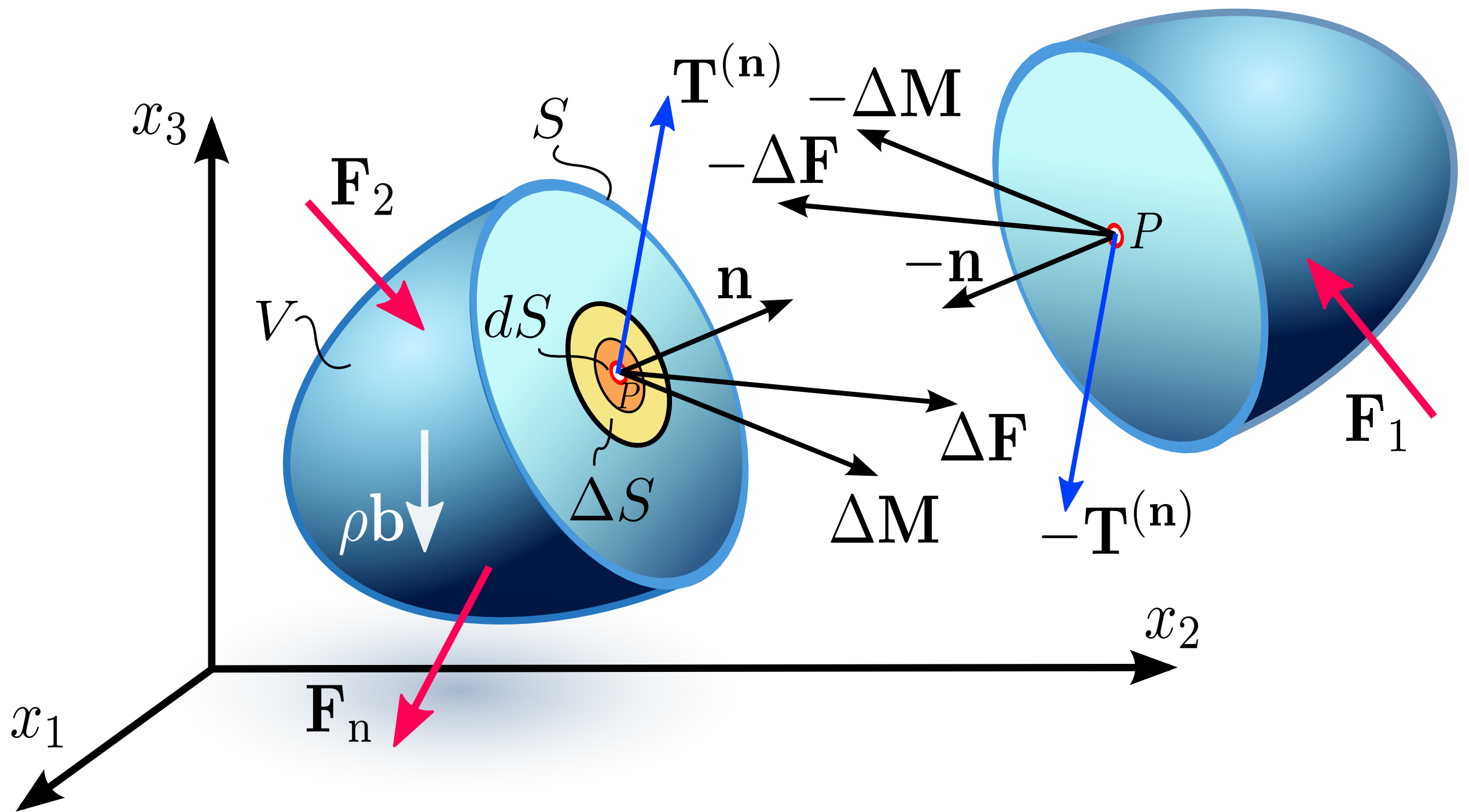
$$\Sigma \mathbf{F} = m \mathbf{a}$$

$$-\frac{\partial p}{\partial y} = m a_y$$

$$-\frac{\partial p}{\partial x} = m a_x$$

$$-\frac{\partial p}{\partial z} - \rho g = m a_z$$

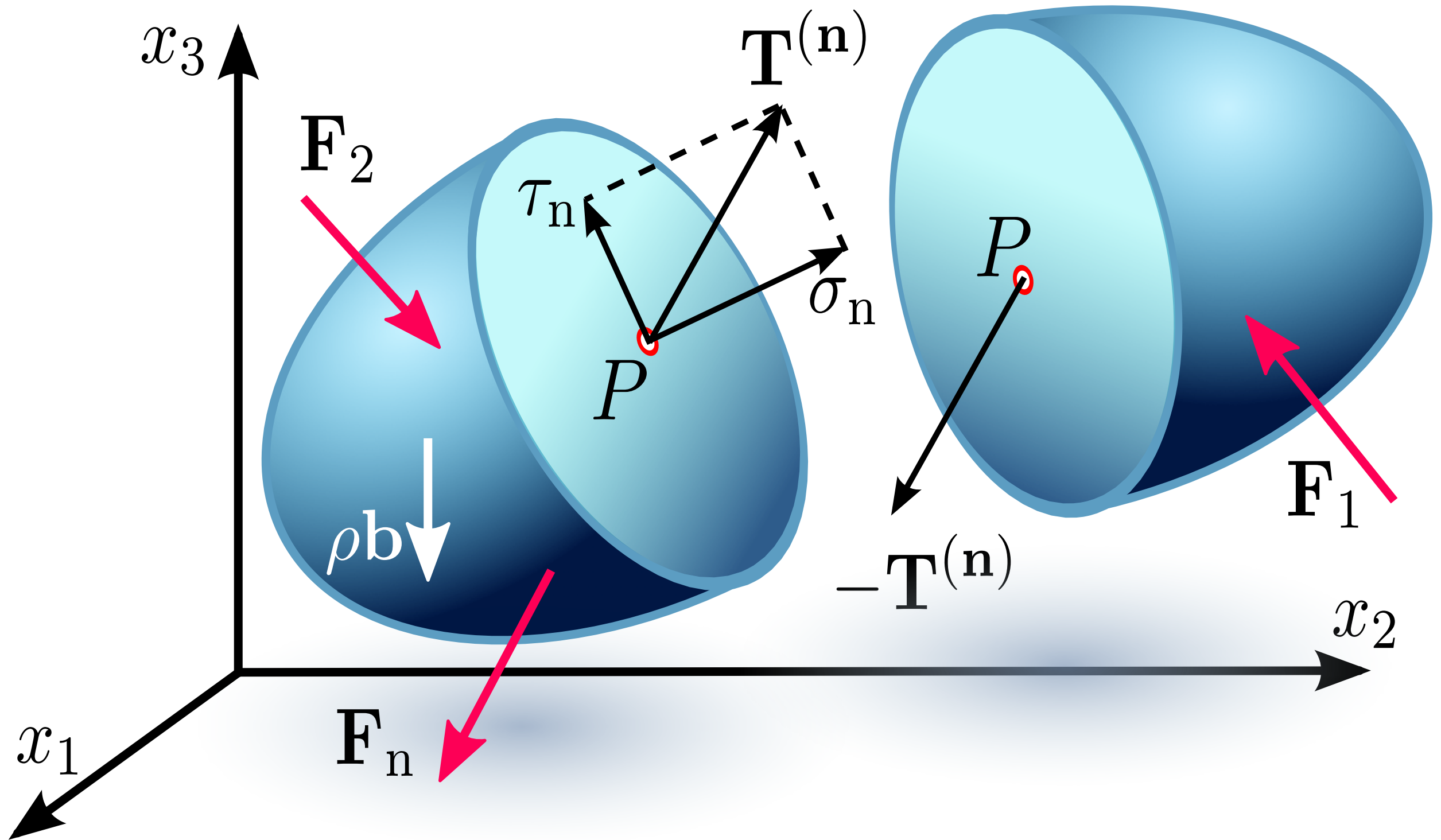
Stress at a point





$$\lim_{\Delta S \rightarrow 0}$$

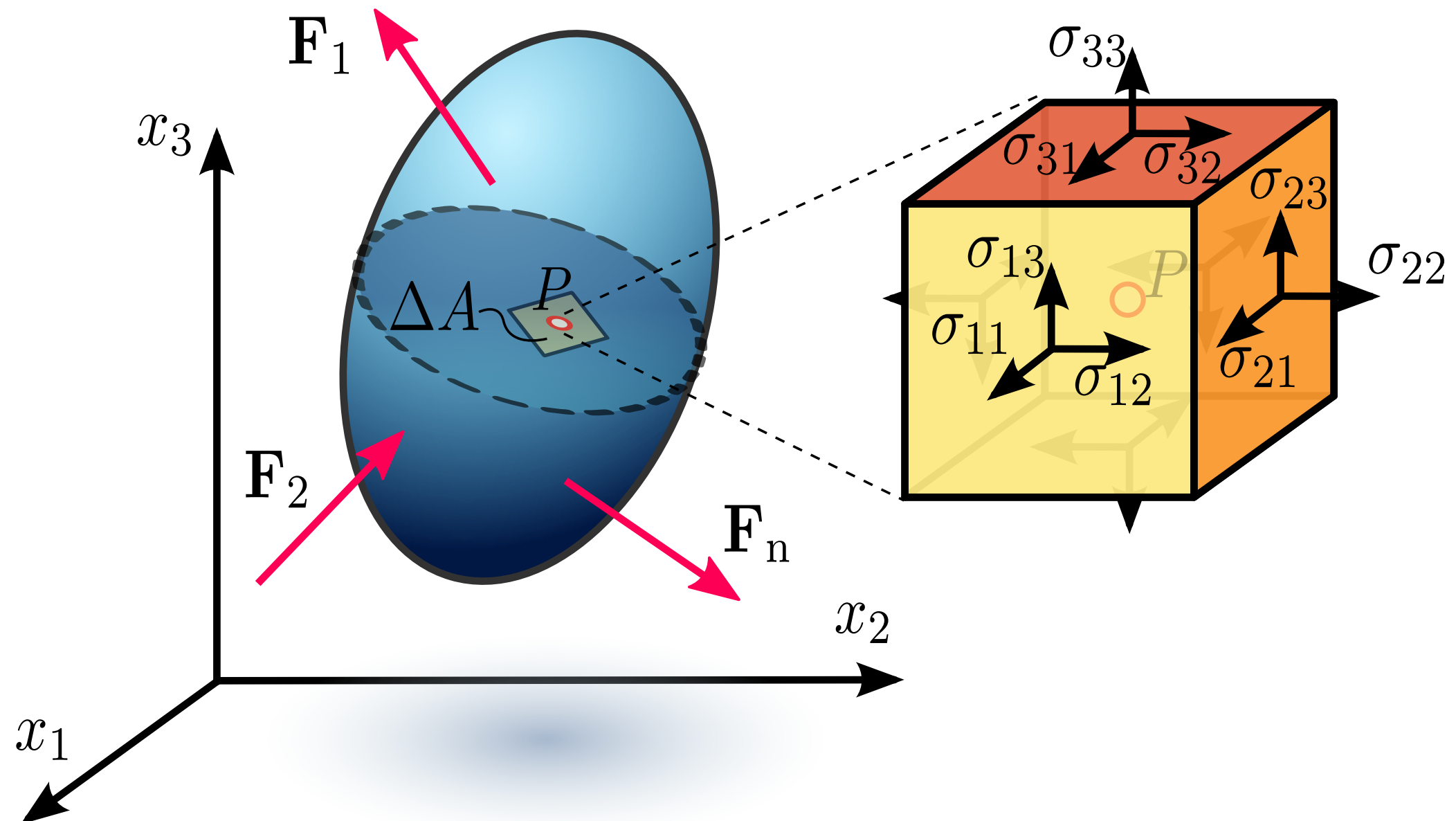
ΔS

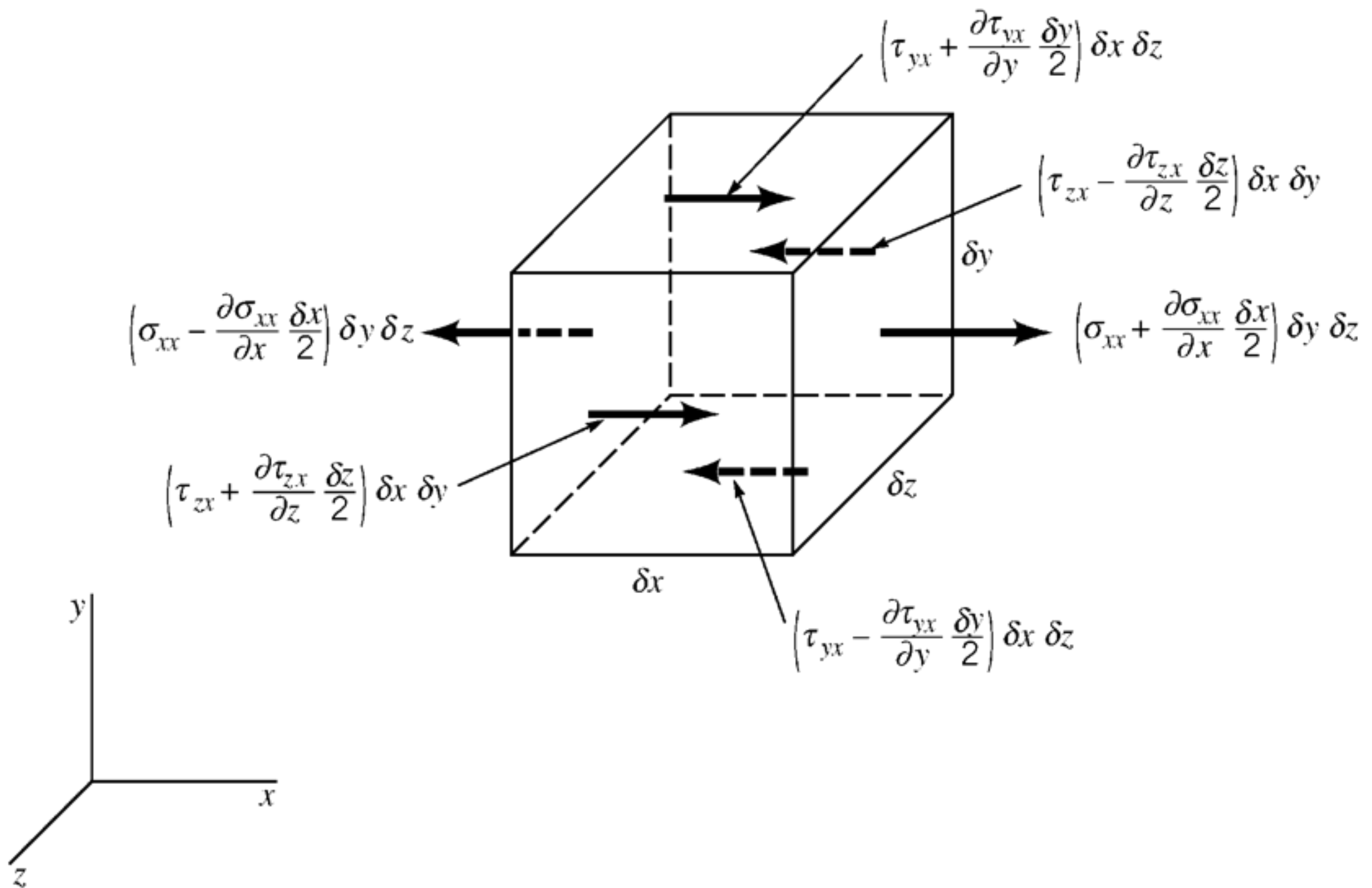


The complete specification of the internal forces requires knowledge of the stress vectors on ALL planes passing through P

Cauchy Stress Theorem

- Knowledge of stress **vector** at three mutually perpendicular planes completes the specification





In General

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} -p + \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & -p + \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & -p + \tau_{zz} \end{bmatrix}$$

$$T_{ij} = -p\delta_{ij} + \tau_{ij}$$

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \mathbf{F}$$

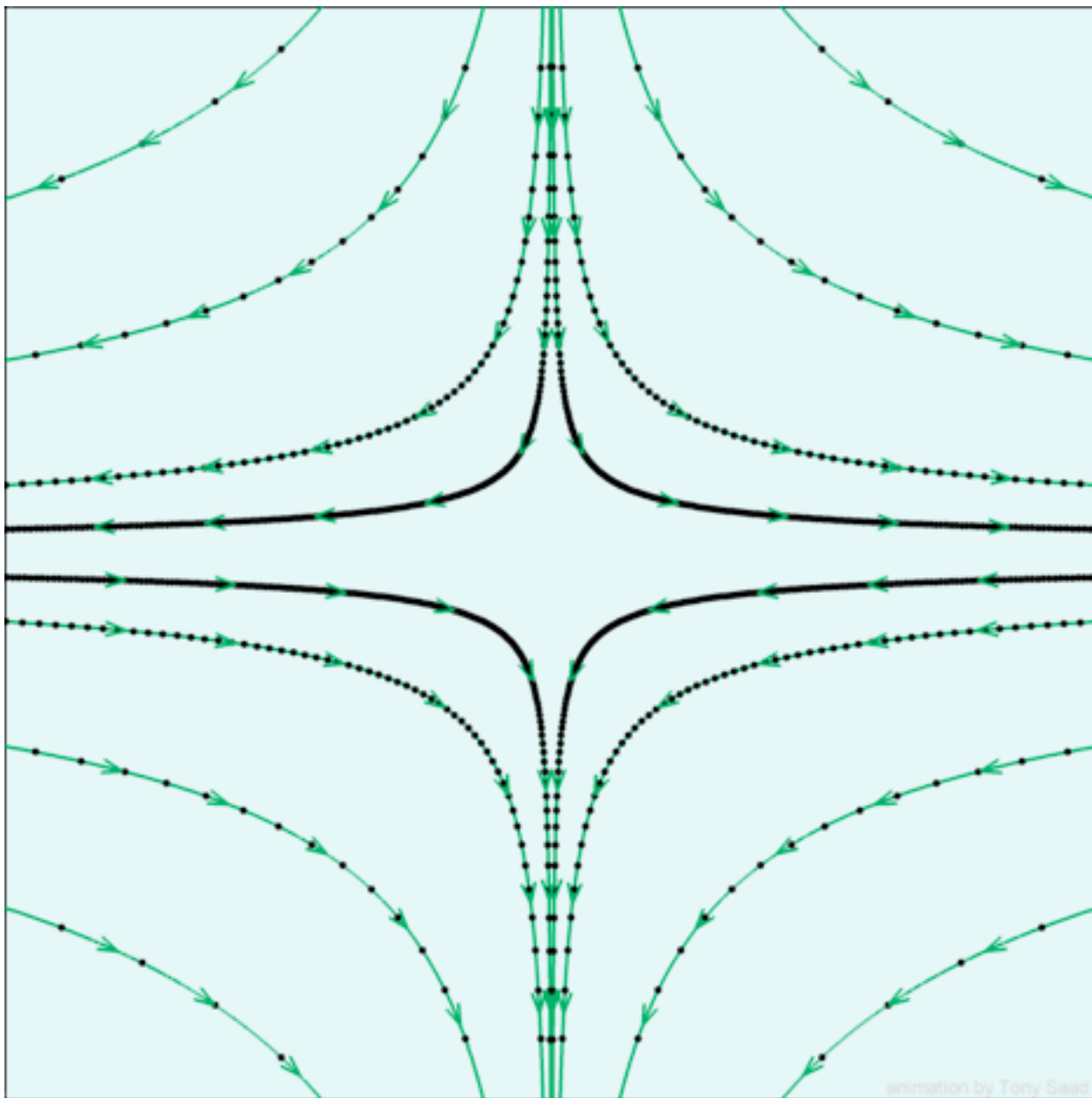
Kinematic implications

$$\nabla \cdot \mathbf{u} = 0$$

$\Downarrow ?$

$$p(x, t)$$

Example



Consider:

$$\mathbf{u} = xy\mathbf{i} - \frac{1}{2}y^2\mathbf{j}$$

Continuity:

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = y - y = 0$$



Assume this velocity field
corresponds to a steady inviscid
flow

$$\nabla p = -\mathbf{u} \cdot \nabla \mathbf{u}$$

Integrate for the Pressure

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial p}{\partial x} = -u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y}$$

$$\frac{\partial p}{\partial y} = -u \frac{\partial v}{\partial x} - v \frac{\partial v}{\partial y}$$

$$\mathbf{u} = xy\mathbf{i} - \frac{1}{2}y^2\mathbf{j}$$

$$\left. \begin{array}{l} \frac{\partial p}{\partial x} = -\frac{1}{2}xy^2 \Rightarrow p(x, y) = -\frac{1}{4}x^2y^2 + c_0 \\ \frac{\partial p}{\partial y} = -\frac{1}{2}y^3 \Rightarrow p(x, y) = -\frac{1}{8}y^4 + c_1 \end{array} \right\} p(x, y) = -\frac{1}{4}x^2y^2 - \frac{1}{8}y^4 + c$$

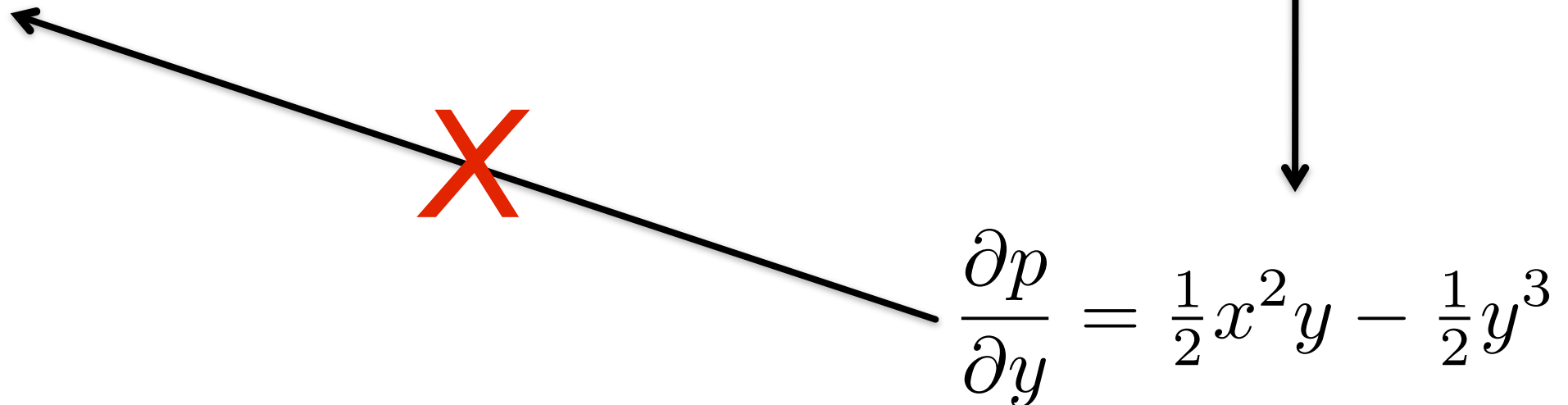
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial p}{\partial x} = -u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y}$$

$$\frac{\partial p}{\partial y} = -u \frac{\partial v}{\partial x} - v \frac{\partial v}{\partial y}$$

$$\mathbf{u} = xy\mathbf{i} - \frac{1}{2}y^2\mathbf{j}$$

$$\left. \begin{array}{l} \frac{\partial p}{\partial x} = -\frac{1}{2}xy^2 \Rightarrow p(x, y) = -\frac{1}{4}x^2y^2 + c_0 \\ \frac{\partial p}{\partial y} = -\frac{1}{2}y^3 \Rightarrow p(x, y) = -\frac{1}{8}y^4 + c_1 \end{array} \right\} p(x, y) = -\frac{1}{4}x^2y^2 - \frac{1}{8}y^4 + c$$



$$\frac{\partial p}{\partial y} = \frac{1}{2}x^2y - \frac{1}{2}y^3$$

What's Wrong?

- Velocity field does not correspond to this model (inviscid)
- Velocity field is unphysical

Clairaut



Differential
Forms
1713 - 1765

d'Alembert



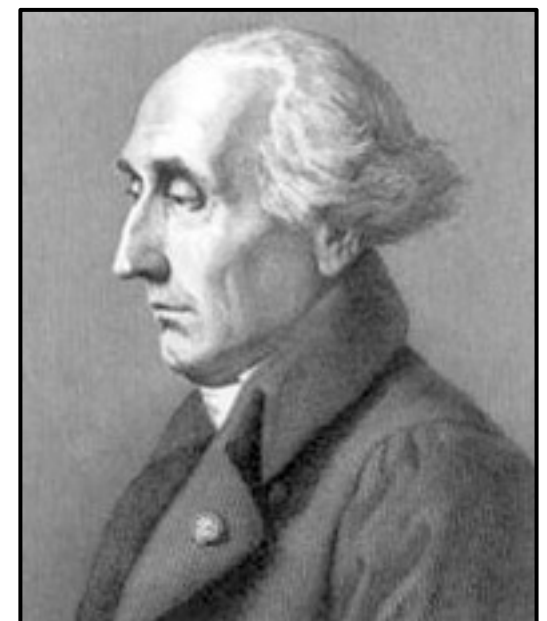
First principles
1717 - 1783

Euler



Synthesis,
Generalization,
Derivations
1707 – present

Lagrange



Solutions
1736 - 1813

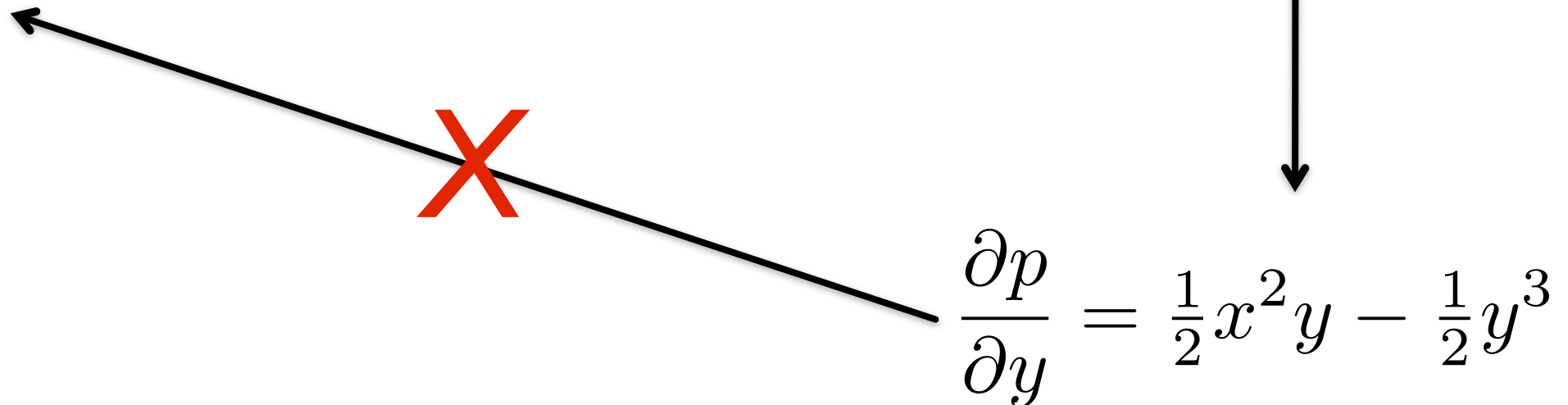
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial p}{\partial x} = -u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y}$$

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$$\mathbf{u} = xy\mathbf{i} - \frac{1}{2}y^2\mathbf{j}$$

$$\left. \begin{array}{l} \frac{\partial p}{\partial x} = -\frac{1}{2}xy^2 \Rightarrow p(x, y) = -\frac{1}{4}x^2y^2 + c_0 \\ \frac{\partial p}{\partial y} = -\frac{1}{2}y^3 \Rightarrow p(x, y) = -\frac{1}{8}y^4 + c_1 \end{array} \right\} p(x, y) = -\frac{1}{4}x^2y^2 - \frac{1}{8}y^4 + c$$



$$\frac{\partial p}{\partial y} = \frac{1}{2}x^2y - \frac{1}{2}y^3$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial p}{\partial x} = -u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y}$$

$$\frac{\partial p}{\partial y} = -u \frac{\partial v}{\partial x} - v \frac{\partial v}{\partial y}$$

$$\mathbf{u} = xy\mathbf{i} - \frac{1}{2}y^2\mathbf{j}$$

$$\frac{\partial p}{\partial x} = -\frac{1}{2}xy^2 \Rightarrow p(x, y) = -\frac{1}{4}x^2y^2 + f(y)$$

$$\frac{\partial p}{\partial y} = -\frac{1}{2}xy^3 = -\frac{1}{2}x^2y + f'(y)$$

$$f'(y) = -\frac{1}{2}y^3 + \frac{1}{2}x^2y$$

FUNCTION OF x AND y !

Enforce f to be a function of y only

$$\frac{\partial}{\partial x} \frac{\partial f}{\partial y} = 0$$

$$\frac{\partial}{\partial x} \frac{\partial f}{\partial y} = \frac{\partial^2 p}{\partial x \partial y} - \frac{\partial^2 p}{\partial y \partial x}$$

$$\frac{\partial^2 p}{\partial x \partial y} = \frac{\partial^2 p}{\partial y \partial x}$$

Integration Constraints

$$\nabla p = -\rho \frac{D\mathbf{u}}{Dt} + \mu \nabla^2 \mathbf{u} \equiv \mathcal{F}_i$$

$$\frac{\partial^2 p}{\partial x_1 \partial x_2} = \frac{\partial^2 p}{\partial x_2 \partial x_1} \quad \frac{\partial^2 p}{\partial x_1 \partial x_3} = \frac{\partial^2 p}{\partial x_3 \partial x_1} \quad \frac{\partial^2 p}{\partial x_2 \partial x_3} = \frac{\partial^2 p}{\partial x_3 \partial x_2}$$

$$\frac{\partial \mathcal{F}_1}{\partial x_1} = \frac{\partial \mathcal{F}_2}{\partial x_2} \quad \frac{\partial \mathcal{F}_1}{\partial x_1} = \frac{\partial \mathcal{F}_3}{\partial x_3} \quad \frac{\partial \mathcal{F}_2}{\partial x_2} = \frac{\partial \mathcal{F}_3}{\partial x_3}$$

Corollary I

- The pressure is integrable iff its total differential is exact

$$dp = \frac{\partial p}{\partial x_1} dx_1 + \frac{\partial p}{\partial x_2} dx_2 + \frac{\partial p}{\partial x_3} dx_3 + \frac{\partial p}{\partial t} dt$$

$$\frac{\partial^2 p}{\partial x_1 \partial x_2} = \frac{\partial^2 p}{\partial x_2 \partial x_1} \quad \frac{\partial^2 p}{\partial x_1 \partial x_3} = \frac{\partial^2 p}{\partial x_3 \partial x_1} \quad \frac{\partial^2 p}{\partial x_2 \partial x_3} = \frac{\partial^2 p}{\partial x_3 \partial x_2}$$

Corollary 2

$$\nabla \times \nabla \phi = \mathbf{0}$$

if and only if ϕ is continuous and twice differentiable

$$\frac{\partial^2 p}{\partial x \partial y} = \frac{\partial^2 p}{\partial y \partial x} \Leftrightarrow \nabla \times \nabla p = \mathbf{0}$$

- The pressure is integrable iff the vorticity transport equation is identically satisfied

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} - \nabla \times \mathbf{u} \times \boldsymbol{\Omega} = -\nu \nabla^2 \boldsymbol{\Omega} \Rightarrow \nabla \times \nabla p = \mathbf{0}$$

Irrotational Motion

$$\Omega = 0$$

$$\nabla \times \nabla p = 0$$

Asymptotic Solutions

$$p = \sum_{n=0}^{\infty} \delta_n(\epsilon) p_n$$

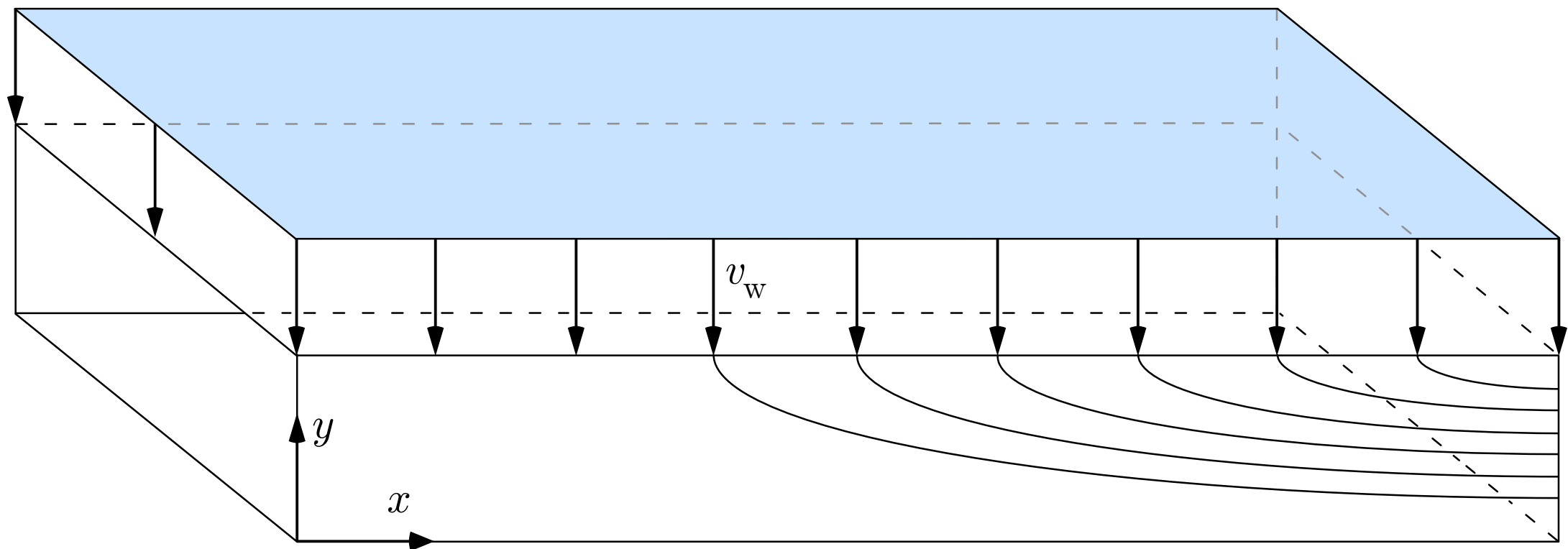
$$\delta_{n+1}(\epsilon) = \mathcal{O}[\delta_n(\epsilon)] ; |\epsilon| < 1$$

$$\frac{\partial^2 p_n}{\partial x_1 \partial x_2} = \frac{\partial^2 p_n}{\partial x_2 \partial x_1} \quad \frac{\partial^2 p_n}{\partial x_1 \partial x_3} = \frac{\partial^2 p_n}{\partial x_3 \partial x_1} \quad \frac{\partial^2 p_n}{\partial x_2 \partial x_3} = \frac{\partial^2 p_n}{\partial x_3 \partial x_2}$$

Applications

Wall Injected Flows

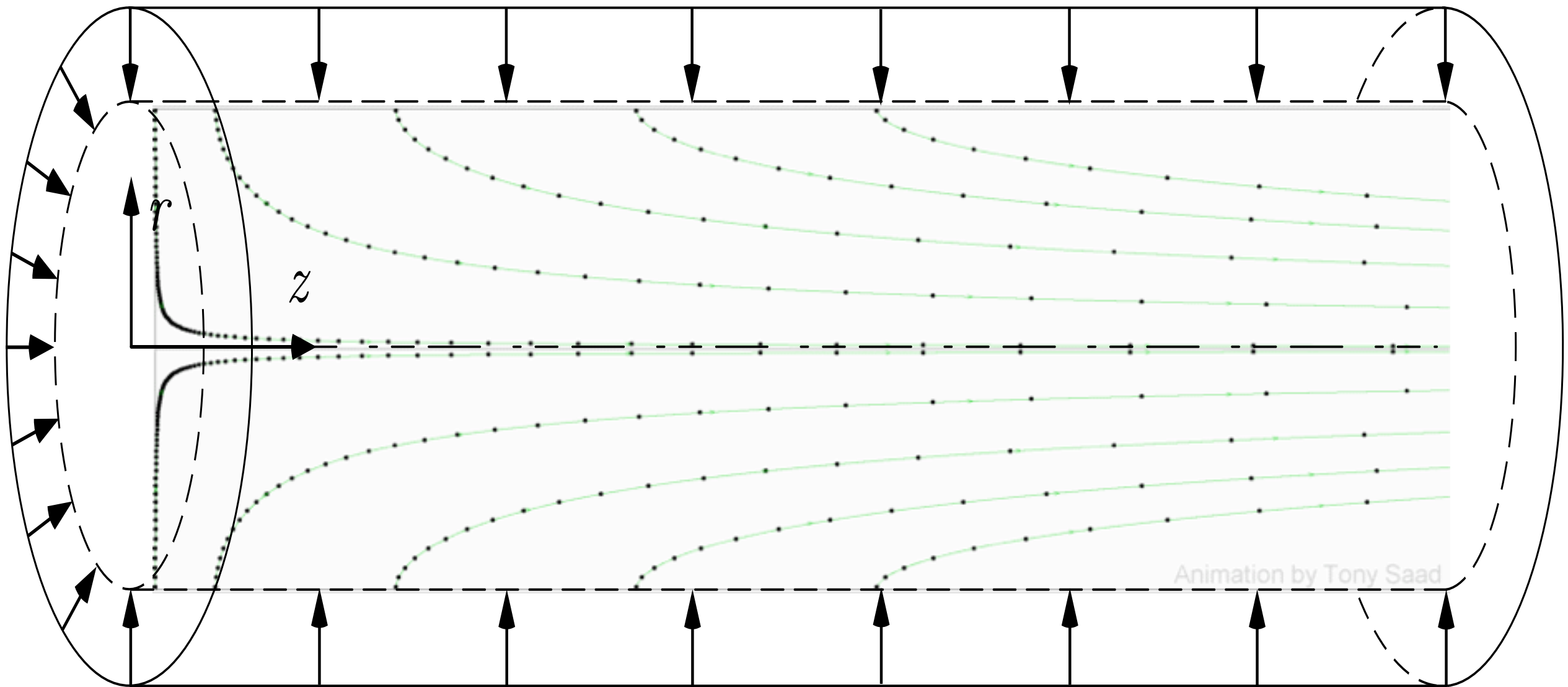
- Injection or suction of a fluid through an arbitrary porous surface

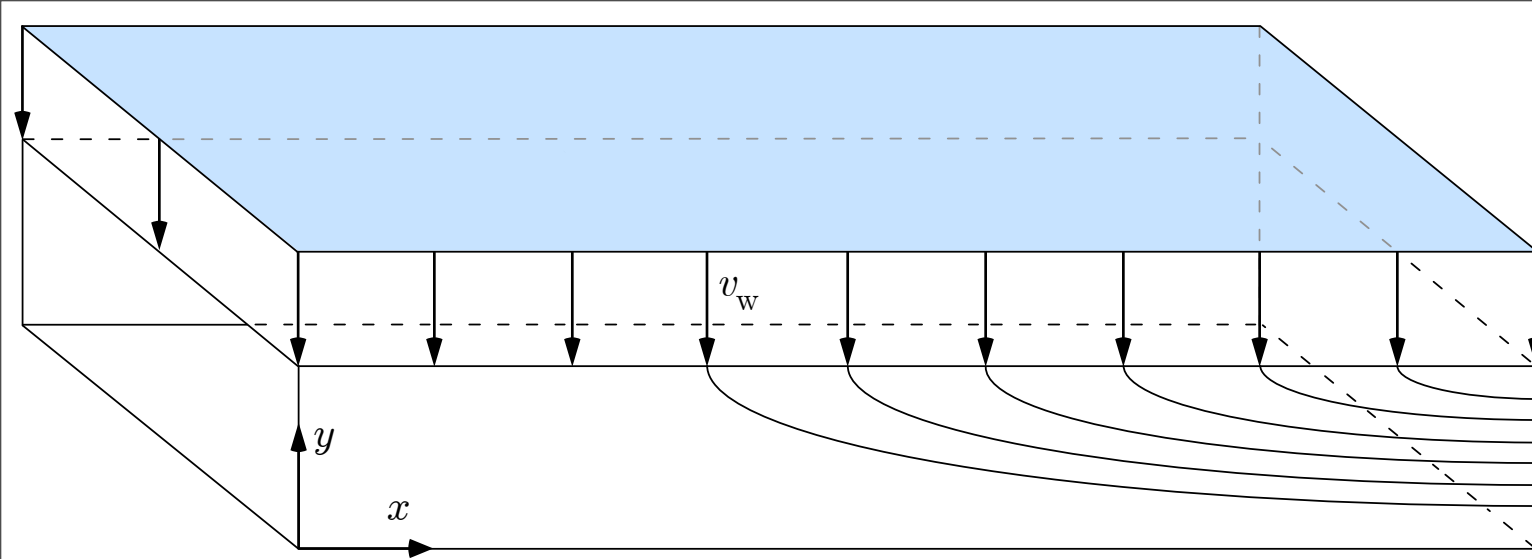


$$\psi = z \sin\left(\frac{1}{2} \pi r^2\right)$$

$$\begin{cases} u_r = -r^{-1} \sin\left(\frac{1}{2} \pi r^2\right) \\ u_z = \pi z \cos\left(\frac{1}{2} \pi r^2\right) \end{cases}$$

- Culick, F. E. C. 1966 Rotational axisymmetric mean flow and damping of acoustic waves in a solid propellant rocket. *AIAA Jl* 4, 1462-1464.
- Taylor, G. I., "Fluid Flow in Regions Bounded by Porous Surfaces," *Proceedings of the Royal Society, London, Series A*, Vol. 234, No. 1199, 1956, pp. 456-475.





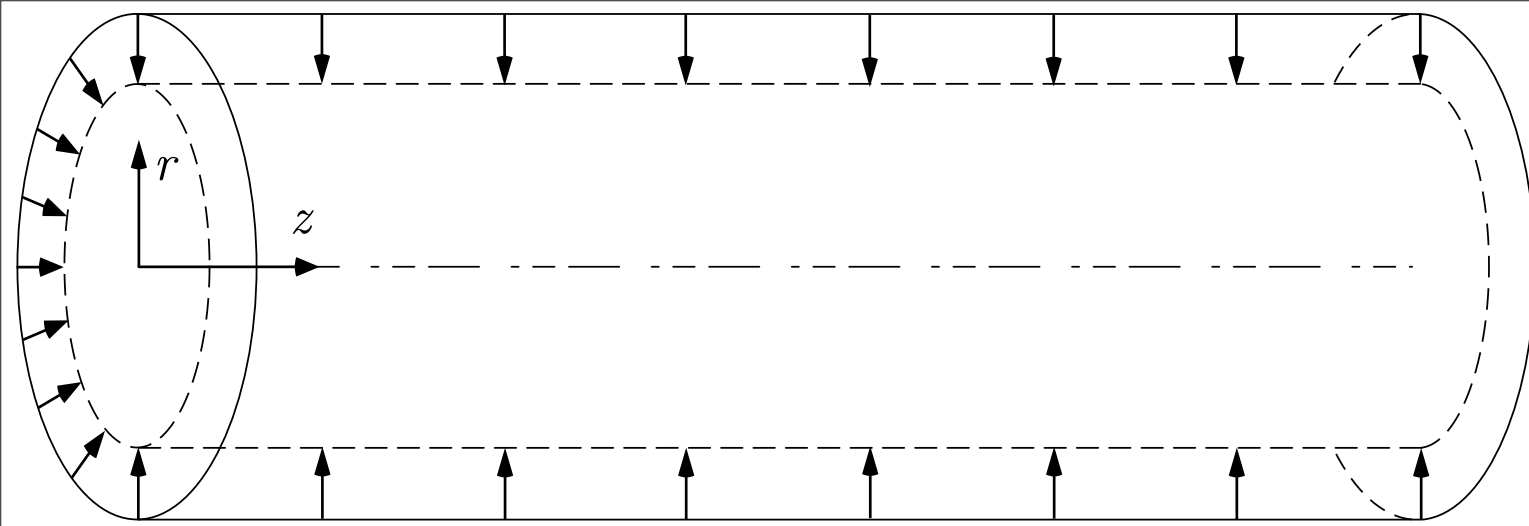
$$\psi = x \sin\left(\frac{1}{2} \pi y\right)$$

$$\begin{cases} u = \frac{1}{2} \pi x \cos\left(\frac{1}{2} \pi y\right) \\ v = -\sin\left(\frac{1}{2} \pi y\right) \end{cases}$$

Constraint:

$$u \frac{\partial^2 u}{\partial x \partial y} + v \frac{\partial^2 u}{\partial y^2} = 0$$

$$p(x, y) = p_0 - \frac{1}{8} \pi^2 x^2 + \frac{1}{4} \cos(\pi y)$$



$$\psi = z \sin\left(\frac{1}{2} \pi r^2\right)$$

$$\begin{cases} u_r = -r^{-1} \sin\left(\frac{1}{2} \pi r^2\right) \\ u_z = \pi z \cos\left(\frac{1}{2} \pi r^2\right) \end{cases}$$

Constraint:

$$u \frac{\partial^2 u_z}{\partial r^2} + u_z \frac{\partial^2 u_z}{\partial r \partial z} - \frac{u_r}{r} \frac{\partial u_z}{\partial r} = 0$$

$$p(r, z) = p_0 - \frac{1}{2} \pi^2 z^2 + \frac{1}{2} r^{-2} \sin^2\left(\frac{1}{2} \pi r^2\right)$$

Arbitrary Injection

$$\psi(x, y) = \underbrace{x \sin\left(\frac{1}{2} \pi y\right)}_{\text{Taylor-Culick}} + \underbrace{\sum_{n=0}^{\infty} \beta_n \sin\left[\left(n + \frac{1}{2}\right) \pi y\right]}_{\text{arbitrary injection}}$$

$$u(x, y) = \frac{1}{2} \pi x \cos\left(\frac{1}{2} \pi y\right) + f(y)$$

$$v(x, y) = -\sin\left(\frac{1}{2} \pi y\right)$$

$$f(y) = \frac{1}{2} \pi \sum_{n=0}^{\infty} (2n+1) \beta_n \cos\left[\left(n + \frac{1}{2}\right) \pi y\right]$$

- Majdalani, J., and Saad, T., “[The Taylor-Culick Profile With Arbitrary Headwall Injection](#),” Physics of Fluids, Vol. 19, No. 9, 2007. ([10.1063/1.2746003](#))
- Saad, T., and Majdalani, J., “[Rotational Flowfields in Porous Channels with Arbitrary Headwall Injection](#),” Journal of Propulsion and Power, vol. 25, No. 4, 2009, pp. 921 - 929. ([10.2514/1.41926](#))

Test for Constraint

$$u \frac{\partial^2 u}{\partial x \partial y} + v \frac{\partial^2 u}{\partial y^2} \neq 0$$

Using this model, the pressure is NOT integrable

- Determined requirement that velocity field must meet for analytic pressure
- Simple constraints can be used to validate a solution
- Kinematic condition (continuity) is necessary, but insufficient for securing a flow
- Vorticity transport is a key equation
- Compressibility effects will be introduced in a future study